Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 4, 1996

ON THE PROPAGATION OF A NONLINEAR-DIFFUSION FRONT

S. S. Titov

UDC 533.6+517.946

For a nonlinear heat conduction (diffusion) equation [1, 2] with power nonlinearity

$$\frac{\partial T}{\partial t} = \operatorname{div}(T^\sigma \operatorname{grad}(T)) + Q(T), \tag{1}$$

it is natural to formulate the problem of perturbation propagation against zero background ($T = 0$) [3–5] and investigate the interrelation of the mechanism of front propagation with the behavior of a nonzero solution in the vicinity of the front. This problem is of both theoretical [movement of zeroes in solutions of nonlinear Eq. (1)] and applied (propagation of a nonlinear-diffusion front) significance. Under the assumption of local analyticity of T, such formulation admits a complete solution in the form of heat-wave propagation [4, 5]. Below, a general solution of this equation in the vicinity of the diffusion front is constructed for rational values of $\sigma > 0$ in the one-dimensional symmetric case. This solution depends on two arbitrary functions t and can contain a singularity of the root type at the point of contiguity with zero background, in contrast to [4], where one arbitrary function was present and contiguity with the background was analytical.

Problem Formulation. For arbitrary σ and source function $Q(T)$, by substituting [4, 6] $u = T^\sigma$ and $t = t/2$, we reduce (1), in the case of symmetry, to the equation

$$u_t = (u^2)_{xx} + \frac{\nu}{x}(u^2)_x + \omega(u_x)^2 + Q_*(u). \tag{2}$$

Here $\omega = 2/\sigma - 2$; $Q_*(u) = 2\sigma u^{(\sigma-1)/\sigma} Q(u^{1/\sigma})$; $\nu = 0$, 1, and 2 for plane, axial, and spherical symmetry, respectively; and x is the spatial variable.

Let $x = h(t)$ be the front of nonlinear diffusion (for $\sigma > 0$, in accordance with [1], the diffusion front exists). With introduction of the variable $\xi = x - h(t)$ Eq. (2) becomes

$$u_t = h_t u_\xi + (u^2)_{\xi\xi} + \omega(u_\xi)^2 + \frac{\nu}{\xi + h}(u^2)_\xi + Q_*(u). \tag{3}$$

We assume that $u = 0$ at the point $\xi = 0$ and construct a solution of the above equation in the vicinity of this point as the series

$$u(\xi, t) = \sum_{n \in P} g_n(t)\xi^n, \tag{4}$$

where $P = \{n_0, n_1, n_2, \ldots\} \subset \mathbb{R}$ $(0 < n_0 < n_1 < n_2 < \ldots)$ is a set of powers of ξ. We wish to determine the structure of series (4), describe the set of its indices P and the coefficients g_n, study series (4) for convergence, and apply it to solution of physically meaningful problems.

Construction of the Solution. The character of contiguity of a solution with the zero background depends on the set P. To determine it, we substitute (4) into (3) for $\nu = 0$, $Q_* = Q = 0$, and obtain the equation

$$\sum_{n \in P} g_n'(t)\xi^n = h'(t) \sum_{k \in P} k g_k(t)\xi^{k-1} + \sum_{m,n \in P} [(m+n)(m+n-1) + \omega mn] g_m(t) g_n(t) \xi^{m+n-2}, \tag{5}$$

Ural Architectural Arts Academy, Ekaterinburg 620219. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 4, pp. 113–118, July–August, 1996. Original article submitted February 27, 1995; revision submitted May 16, 1995.

 0021-8944/96/3704-0554 $15.00

in which the nonlinear term allows for introduction [7] of some multiplication $(\cdot)$ for a set of functions that depend on ξ so that

$$v \cdot w = (vw)_{\xi\xi} + \omega v_\xi w_\xi, \tag{6}$$

or for the subalgebra generated by the power functions

$$\xi^m \cdot \xi^n = [(m+n)(m+n-1) + \omega mn]\xi^{m+n-2}. \tag{7}$$

It follows from (5) that to obtain a closed system of equations for $h(t)$ and the coefficients $g_n(t)$, it will suffice to require that the set $B = \{\xi^{n_0}, \xi^{n_1}, \xi^{n_2}, \ldots\}$ be closed relative to multiplication (7) and the shift $\xi^k \mapsto \xi^{k-1}$, i.e., require that B be a subset of the basis of a certain ring (algebra) with multiplication (7) and $\xi^1 \in B$.

The structure of algebra (6) and (7) is rather simple: it is graduated so that $\xi^{(\alpha-2)} \cdot \xi^{(\beta-2)} = \lambda_{\alpha,\beta}(\omega)\xi^{(\alpha+\beta)-2}$, where $\lambda_{\alpha,\beta}(\omega) \neq 0$ everywhere, except for a certain second-order curve (depending on ω) in the plane $\alpha O \beta$; in particular, there are two nilpotent elements ε (for which $\varepsilon \cdot \varepsilon = 0$) in this algebra:

$$\varepsilon_1 = \xi^0 \qquad \text{and} \qquad \varepsilon_2 = \xi^\delta$$

$[\delta = 2/(\omega+4)$, if $\omega \neq -4$ (i.e., $\sigma \neq -1$)]. Since we are constructing a solution that is contiguous with zero, $\varepsilon_1 = \xi^0 = 1 \notin P$. At the same time, if $P \supseteq \mathbb{N}$ does not contain any nilpotent elements, then, by virtue of (5), the smallest noninteger power of ξ^α in solution (4) is necessarily such that $\xi^\alpha \cdot \xi^\alpha = 0$ in terms of multiplication (7), i.e., $\alpha = \delta$. Thus, if $\delta \notin P$, the solution is analytical in ξ, i.e., $P = \mathbb{N}$, and we obtain a regular case which was investigated earlier [2–5]. To extend it, we shall assume nonanalyticity of the solution of u for $u = 0$.

For $h'(t) \neq 0$ and rational $\sigma = -1 + p/q$ $(p, q \in \mathbb{N})$, we have

$$\omega = 2\frac{2q-p}{p-q}, \qquad \delta = 1 - \frac{q}{p} = \frac{p-q}{p}.$$

Ignoring the possibility that $\lambda_{\alpha,\beta} = 0$, we represent algebra (7) as a homomorphic image of the group algebra for the additive group of rational numbers $\mathbb{Q}$. This leads to the following description of the set of indices P for $\sigma > 0$: since $n_0 = \delta = 2/(\omega+4) \in P$, $1 = p/p \in P$, and p and q are coprime, the set P is embedded in an arithmetical progression with a difference $1/p$ and the smallest element n_0. Hence, we have $n_i = \delta + (1/p)i$, $i \in \mathbb{N}$.

However, for noninteger $\sigma > 0$, it is not trivial to verify that such series actually satisfies the equation being solved. Nevertheless, recurrence calculation of the coefficients in the specified numbering for rational σ $(\sigma \in \mathbb{Q}$, $\sigma = p/q - 1 > 0)$ gives a representation of the solution of Eq. (3) as a power series in the variable $\xi^{1/p}$:

$$u(\xi, t) = G(t)\xi^{1-q/p} + H(t)\xi + \sum_{n=p+1}^{\infty} g_{n/p}(t)\xi^{n/p}, \tag{8}$$

i.e., $g_{k/p} = 0$ for $k < p + q$, except for $k = p$ and $k = p - q$. Here $G(t)$ is an arbitrary function which does not vanish in the vicinity considered; $H(t) = g_1(t)$ is an arbitrary function which is proportional to the velocity $h'(t)$ of diffusion-front propagation:

$$h'(t) = -2\,\frac{q(p+q)}{p(p-q)}H(t).$$

Moving to integer numbering for brevity and setting $G_n = g_{n/p}$ so that $G = G_{p-q}$, $H = G_p$, etc., for any ν, we have

$$G_{p+q}(t) = \frac{pq}{2(p^2 + pq - 2q^2)}\,\frac{[H(t)]^2}{G(t)},$$

and the coefficients of the series are determined recursively by the formula

$$G_{m+p+q}(t) = \frac{1}{2(m+p+2q)(m+2q)G(t)} \left\{ p^2 \frac{G'_m}{G(t)} - 2(m+p)(m+q)\frac{H(t)}{G(t)} G_{m+p}(t) \right.$$

$$- \sum_{k=p+q}^{m+p-q} \frac{(m+2p)(m+p)(p-q) + 2(2q-p)k(m+2p-k)}{p-q} G_k G_{m+2p-k}$$

$$\left. - p \sum_{s \geqslant 0} \frac{\nu(-1)^s}{h^{s+1}} \sum_{k=p-q}^{m+q-ps} [m + (1-s)p] G_k G_{m+p-k-ps} \right\}. \qquad (9)$$

Thus, the solution of Eq. (3) for rational $\sigma > 0$ can be constructed in the form of (8) and not only for $\nu = 0$ and $Q = 0$, because the last two terms in (3) do not influence isolation of singularities: the source function $Q(u)$ is assumed to be analytical, $Q(0) = 0$, and the factor $\nu/(h+\xi)$ is expanded (as a geometrical progression) in a series in integer powers of ξ for $h(t) \neq 0$. Therefore, inclusion of the source $Q \not\equiv 0$ adds to the right-hand side of (9) nonlinearity due to the calculated coefficients, which is similar to the last nonlinearity in (9) taking into account the term with $\nu \neq 0$. This expression is not given here because it is cumbersome.

Structure of the Series. To attain computational efficiency, it is necessary to investigate the structure of the series derived.

Using algebra (6) and (7), and recursion (9), we write the coefficients G_n as the differential polynomials

$$G_n(t) = \sum_{m=\{m_k\}, l=\{l_k\}} U_{m,l} \prod_{k \geqslant -1} [G^{(k)}(t)]^{m_k} [H^{(k)}(t)]^{l_k},$$

where U_{ml} are numerical factors that depend on the sets of powers $m = \{m_k\}$ and $l = \{l_k\}$; $H^{(-1)} \sim h$.

Lemma 1. *The powers of derivatives for any coefficient $G_{n/p}$ satisfy the relation*

$$\sum_{k \geqslant -1} \left\{ \left[\frac{p-q}{p} + 2k \right] m_k + (1+2k) l_k \right\} = \frac{n}{p}.$$

The given lemma reduces algebraic recursion (9) to a numerical recursion for the factors $U_{m,l}$. This statement can be expressed as an analytical formulation which represents a heat wave as a nonlinear superposition of the processes of front propagation and heat transfer in its vicinity.

Lemma 2. *Each differential monomial in G_n coincides with accuracy up to a numerical multiplier with the differential monomial in the expression*

$$\left(\frac{1}{G} \frac{\partial}{\partial t} \right)^S \left[\frac{H^{\lambda+1}}{G^\lambda} \right]$$

for the representation of n in the form $n = (1+S)p + (\lambda + S)q$, where λ and S are integer numbers ($S \geqslant 0$ and $\lambda \geqslant -1$).

Investigation of the structure of the series and its summation for polynomial $G(t)$ and $h(t)$ gives a representation of the solution in the form of an analytical function of ξ and a finite number of arguments, i.e., derivatives of $G(t)$ and $h(t)$, in spite of the fact that the Lie–Becklund group [6] of Eq. (1) for $\sigma > 0$ is trivial [8–10].

Convergence of the Series. Taking into account Lemmas 1 and 2, using the substitution $y = \xi^{1/p}$ and $z = T^{p/q}/\xi$, the convergence of the series follows from an analog of the Kowalewski theorem [4] for $z|_{y=0} \equiv G_{p-q}(t) \equiv G \neq 0$, since, by virtue of the foregoing, the formal series for the solution exists for any functions $G_0(t)$ and $h(t)$ that are analytical in the vicinity of the point $(y, t) = (0, t_0)$.

Theorem. *For any functions $G(t)$, $H(t)$, and $G(t_0) \neq 0$ that are analytical in the vicinity of the point $t = t_0$, any $\sigma = p/q - 1 > 0$, any $\nu \geqslant 0$, and any source function Q $[Q(0) = 0]$ analytical for zero, the series (8) with recursion (9) is a solution of Eq. (3) that is analytical in the vicinity of the point $t = t_0$, $\xi = 0$, which represents, on the strength of (2) and (3), a solution to Eq. (1) that is contiguous with zero.*

It should be noted that the condition $G(t) \neq 0$ separates the set of the solutions constructed from the solutions considered in [2–5]. The specific case $G(t_0) = 0$ requires further investigation.

Propagation of a Heat Wave in a Plasma. As an example, let us consider Eq. (1) for $p/q = 7/2$, which corresponds to a heat wave in a plasma [11], $\sigma = 5/2$.

The construction presented above gives

$$u(\xi, t) = G(t)\xi^{5/7} + H(t)\xi + g_{9/7}\xi^{9/7} + \sum_{n=11}^{\infty} g_{n/7}(t)\xi^{n/7},$$

where, after the renumbering $G_n = g_{n/7}$, we find sequentially that

$$g_1 = G_7 = H(t) = -35h'/36, \quad G_8 = 0, \quad G_9 = \frac{7}{55}\frac{H^2}{G}, \quad G_{10} = 0, \quad G_{11} = -\frac{42}{715}\frac{H^3}{G^2},$$

$$G_{12} = -\frac{5}{14}\nu\frac{G}{h}, \quad G_{13} = -\frac{25074}{983125}\frac{H^4}{G^3}, \quad G_{14} = g_2 = \frac{49}{288}\frac{G'}{G} - \frac{3}{8}\nu\frac{H}{h}, \quad G_{15} = -\frac{17262}{3342625}\frac{H^5}{G^4}$$

etc. In the case of $h'(t) \equiv 0$, the construction gives a solution of the physical problem of a stopped heat wave. Here, in the line of the stopped front $h(t) \equiv x_0$, we have one arbitrary analytical function at our disposal, i.e., the heat flux from the boundary, which is determined by the function $G(t)$.

This arbitrary function can be used to investigate the temperature regime near the axis of symmetry [12] and also to formulate and solve the generalized Stefan problem with a given heat flux at the wave front.

The convergence of a heat wave to the center of symmetry at a constant speed is also an important model example. If the heat flux at the wave front is constant, then, in the case of plane symmetry ($\nu = 0$), we obtain a solution of the type of a traveling wave, which is not suitable for $\nu \neq 0$. For these values of ν, the method considered above gives a solution in the form of a series that can be used, in particular, to formulate a boundary regime that initiates a wave [2].

Thus, specifying $h(t) = x_0 - vt$, $v = \text{const} > 0$ for a front that propagates uniformly to the center, we find that the boundary temperature regime at the initial radius $x = x_0$ must have the form

$$T\Big|_{x=x_0} = G^{2/5}v^{2/7}t^{2/7}\Big\{1 + \frac{35}{36}\frac{v^{9/7}}{G}t^{2/7} + \frac{1715}{156816}\frac{v^{16/7}}{G^2}t^{4/7}\frac{60025}{1111968}\frac{v^{27/7}}{G^3}t^{6/7} - \frac{5}{14}\nu\frac{1}{x_0}vt + O(t^{8/7})\Big\}^{2/5}.$$

The author is grateful to A. F. Sidorov for his interest in the study and to the reviewer for constructive remarks.

REFERENCES

1. O. A. Oleinik, A. S. Kalashnikov, and Chzhou Yui Lin', "Cauchy and boundary-value problems for an equation of the nonstationary-filtration type," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **22**, No. 5, 667–704 (1958).
2. A. F. Sidorov, "Analytical representations of solutions to nonlinear parabolic equations of the nonstationary-filtration type," *Dokl. Akad. Nauk SSSR*, **280**, No. 1, 47–51 (1985).
3. A. F. Sidorov, "Some analytical representations of solutions to the nonlinear equation of nonstationary filtration for a multiphase incompressible fluid," in: *Numerical Methods of Continuum Mechanics* [in Russian], Inst. of Theor. and Appl. Mech., Sib. Div., Acad. of Sci. of the USSR, **15**, No. 2 (1984), pp. 121–133.
4. S. P. Bautin, "Use of characteristic series to represent solutions of nonlinear equations of parabolic type in the vicinity of a degeneracy line," in: *Numerical Methods of Continuum Mechanics* [in Russian], Inst. of Theor. and Appl. Mech., Sib. Div., Acad. of Sci. of the USSR, **16**, No. 5 (1985), pp. 16–28.
5. A. F. Sidorov, "Use of special constructions of series to calculate singularities of generalized solutions to nonlinear equations," in: *Construction of Algorithms and Solutions to the Problems of Mathematical Physics* (Collected scientific papers), Inst. of Appl. Math., Acad. of Sci. of the USSR (1987), pp. 79-97.
6. N. Kh. Ibragimov, *Groups of Transformations in Mathematical Physics* [in Russian], Nauka, Moscow (1983).

7. S. S. Titov, "The method of finite-dimensional rings for solution of nonlinear equations of mathematical physics," in: *Aerodynamics*, Saratov Univ., Saratov (1988), pp. 104–110.
8. L. V. Ovsyannikov, "Group properties of a nonlinear heat conduction equation," *Dokl. Akad. Nauk SSSR*, **125**, No. 3, 492–495 (1959).
9. V. A. Dorodnitsyn and S. R. Svirshchevskii, " On the Lie–Becklund groups admitted by a heat conduction equation with a source," Preprint No. 101, Inst. Appl. Math., Acad. of Sci. of the USSR, Moscow (1983).
10. S. S. Titov, "On the type of nonlinearity in a symmetric heat conduction equation with a nontrivial Lie–Becklund group," in: *Modern Group Analysis*, Preprint No. 116, Leningrad Inst. of Informatics and Automation, Acad. of Sci. of the USSR, Leningrad (1990).
11. V. S. Imshennik, "Cumulation of converging shock waves with allowance for dissipative processes," *Zh. Prikl. Mekh. Tekh. Fiz.*, No. 6, 10–19 (1980).
12. S. S. Titov, "Representation of solutions to a nonlinear axisymmetric equation of gas filtration in the form of a logarithmic series," in: *Dynamics of Continuous Media* [in Russian], Institute of Hydrodynamics, Novosibirsk, **68** (1984), pp. 132–144.